Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 9

Problem 33 (4 points)**.** Let d, e, f be F-codes.

(a) Show that the following statements are absolute between transitive models M of ZFC:

$$x \in d^M, \quad d^M \neq \emptyset, \quad d^M \subseteq e^M, \quad x \in \mathbb{R} \setminus d^M, \quad d^M = e^M \cap f^M.$$

(b) Find a property of F-codes which is not absolute between transitive models of ZFC.

Suppose that S is an uncountable set and $\kappa > \omega$ is a cardinal. Suppose that $A \subseteq [S]^{<\kappa} = \{X \subseteq S \mid |X| < \kappa\}$ or $A \subseteq [S]^{\kappa} = \{X \subseteq S \mid |X| = \kappa\}$.

(1) A is *unbounded* if for all $x \in [S]^{<\kappa}$ (or $[S]^{\kappa}$) there is $y \in A$ with $x \subseteq y$.

(2) A is *closed* if for all $\subseteq$-*chains* $\langle x_\alpha \mid \alpha < \gamma \rangle$ of sets in A, i.e. $x_\alpha \subseteq x_\beta$ for $\alpha < \beta$, with $\gamma < \kappa$ (resp. $\gamma \leq \kappa$), $\bigcup_{\alpha<\gamma} x_\alpha \in A$.

(3) A is *stationary* if $A \cap C \neq \emptyset$ for every *club* (closed unbounded) $C \subseteq [S]^{<\kappa}$ (or $[S]^{\kappa}$).

(4) A is *directed* if for all $x, y \in A$ there is $z \in A$ such that $x \cup y \subseteq z$.

Problem 34 (6 points)**.** If $f : [S]^{<\omega} \to [S]^{<\kappa}$ then we define

$$C_f = \{x \in [S]^{<\kappa} \mid \forall e \in [x]^{<\omega}(f(e) \subseteq x)\}$$

the set of *closure points of f*.

(a) Suppose that $C \subseteq [S]^{<\kappa}$ is closed and $A \subseteq C$ is directed with $|A| < \kappa$. Show that $\bigcup A \in C$.

(b) Show that for every club subset of $[S]^{<\kappa}$ there is $f : [S]^{<\omega} \to [S]^{<\kappa}$ such that $C_f \subseteq C$.

(c) Show that for every function $f : [S]^{<\omega} \to [S]^{<\kappa}$, C_f is club in $[S]^{<\kappa}$.

Hint for (b): Construct f by induction on $|e|$ such that $f(e) \in C$ with $e \subseteq f(e)$ and $f(d) \subseteq f(e)$ for $d \subseteq e$.

Problem 35 (6 Points)**.** Suppose that $\kappa \leq \lambda \leq \mu$ are uncountable regular cardinals. For $Y \subseteq [\mu]^{<\kappa}$, the *projection* of Y to λ is defined as

$$Y_\lambda = \{y \cap \lambda \mid y \in Y\}.$$

For $X \subseteq [\lambda]^{<\kappa}$, the *lifting* of X to μ is defined as

$$X^\mu = \{x \in [\mu]^{<\kappa} \mid x \cap \lambda \in X\}.$$

Prove the following statements:

(a) If S is stationary in $[\mu]^{<\kappa}$, then S_λ is stationary in $[\lambda]^{<\kappa}$.
(b) If C is club in $[\mu]^{<\kappa}$, then C_λ contains a club in $[\lambda]^{<\kappa}$.
(c) If S is stationary in $[\lambda]^{<\kappa}$, then S^μ is stationary in $[\mu]^{<\kappa}$.

Hint for (b): Use Problem 34.

A forcing $\mathbb{P}$ is said to be *proper*, if for every uncountable cardinal κ, every stationary subset of $[\kappa]^\omega$ remains stationary in every $\mathbb{P}$-generic extension of M.

Problem 36 (4 points)**.** Suppose that $\mathbb{P}$ is proper and G is M-generic for $\mathbb{P}$.

(a) Show that for countable every set $X \in M[G]$ of ordinals there is a set $Y \in M$ such that Y is countable in M and $X \subseteq Y$.
(b) Conclude that $\mathbb{P}$ preserves $\aleph_1$.

Please hand in your solutions on Monday, 18.01.2015 before the lecture.